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ELECTRIC-MAGNETIC DUALITY AND THE "LOOP REPRESENTATION" IN ABELIAN GAUGE THEORIES

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Abstract

Abelian Gauge Theories are quantized in a geometric representation that generalizes the Loop Representation and treates electric and magnetic operators on the same footing. The usual canonical algebra is turned into a topological algebra of non local operators that resembles the order-disorder dual algebra of 't Hooft. These dual operators provide a complete description of the physical phase space of the theories.

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Dual symmetry in Electromagnetism has been a source of recent developments in Theoretical Physics. Besides the related dual symmetry in superstrings theory (for a recent review and references see ref 1), the notion of duality appears in supersymmetric Yang-Mills and topological theories [2-6]. There is also a renewed interest in the study of the analogous to electric-magnetic duality in conventional Yang-Mills theory [7-11].

The purpose of this letter is to study, within the geometrical language that the Loop Representation (L.R.) provides [12-15], the topological content underlying the "electro-magnetic" duality that ordinary Abelian gauge theories present. As we shall see, when these theories are quantized in a generalized L.R. that explicitly incorporates the dual symmetry, the canonical Poisson algebra results translated into a non-local algebra of geometric dependent operators. This algebra is topological, in the sense that it is not affected by continuous deformations of the spatial manifold.

To begin, let us consider free Maxwell theory in 3+1 dimensions. We shall work in flat space time, with $g_{\mu\nu} = diag(+,-,-,-)$. Maxwell equations

$$\partial^{\mu} F_{\mu\nu} = 0 \tag{1.a}$$

$$\partial^{\mu}\ {}^{*}F_{\mu\nu} = 0 \tag{1.b}$$

where ${}^{*}F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$, are invariant under $\vec{E} \rightarrow -\vec{B}, \vec{B} \rightarrow \vec{E}$ (i.e.: $F \rightarrow {}^{*}F$). Then, it is equally admisible to take for the Maxwell lagrangian, the usual one:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}(A) F^{\mu\nu}(A) \tag{2}$$

with $F_{\mu\nu}(A) = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, or a "dual" lagrangian

$$\widetilde{\mathcal{L}} = -\frac{1}{4}\ {}^{*}F_{\mu\nu}(\widetilde{A})\ {}^{*}F^{\mu\nu}(\widetilde{A}) \tag{3}$$

where ${}^*F_{\mu\nu}(\widetilde{A}) = \partial_\mu \widetilde{A}_\nu - \partial_\nu \widetilde{A}_\mu$.

Since both A and $\widetilde{A}$ are one-forms obeying the same dinamical equations, it is often said that Maxwell theory is "self-dual". From the canonical point of view, "self-duality" states that the weak and strong coupling regimes of the theory may be exchanged. In general abelian theories (even in electromagnetism when $d \neq 4$), it may hapen that duality transformation maps one theory into another different one. For instance, as it is well known, the massless scalar theory is dual to the second rank gauge theory in $d = 4$.

To some extent, both the "direct" and "dual" descriptions of Maxwell field can be considered from the begining, by taking the first order master lagrangian.

$$\mathcal{L}' = \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}\partial_\mu A_\nu B_{\lambda\rho} - \frac{1}{4}B_{\lambda\rho}B^{\lambda\rho} \tag{4}$$

Varying B and A produces the equations of motion

$$\varepsilon^{\mu\nu\lambda\rho}\partial_\mu A_\nu = B^{\lambda\rho} \tag{5}$$

and

$$\varepsilon^{\mu\nu\lambda\rho}\partial_\mu B_{\lambda\rho} = 0 \tag{6}$$

respectively. Replacing $B^{\lambda\rho}$ from (5) into (4), gives back the usual Maxwell lagrangian (2). On the other hand, solving equation (8) yields:

$$B_{\lambda\rho} = \partial_\lambda \widetilde{A}_\rho - \partial_\rho \widehat{A}_\lambda \tag{7}$$

whenever the space-time manifold is "topologically trivial" (more precisely, when the 2^{nd} De Rham Cohomology group is trivial). Inserting (7) into (4) produces the "dual" lagrangian $\widetilde{\mathcal{L}}$.

When performing the canonical analysis from $\mathcal{L}'$, however, one encounters an undesirable feature from the point of view of the L.R. approach. Since the B^2 term breaks the evident

gauge invariance (see equation (10)) that the first one has, one is lead to deal with second class constraints. To surmount this difficulty, one can add an auxiliary field C_μ, and consider the modified lagrangian:

$$\mathcal{L}'' = \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}\partial_\mu A_\nu B_{\lambda\rho} - \frac{1}{4}(B_{\lambda\rho} + C_{\rho,\lambda} - C_{\lambda,\rho})^2 \tag{8}$$

which is invariant under the gauge transformations

$$\delta A_\mu = \wedge_{,\mu} \tag{9}$$

$$\delta B_{\lambda\rho} = -\xi_{\rho,\lambda} + \xi_{\lambda,\rho} \tag{10}$$

$$\delta C_\rho = \xi_\rho \tag{11}$$

The issue of adding an (unphysical) auxiliary field to turn the constraints into first class, has been used in the past [16-17] in several contexts.

Once we have a full gauge-invariant theory, we can proceed to a "L.R. formulation". A lot of work is saved following Fadeev and Jackiw [18] by noticing that from the very structure of the "B-F" term:

$$\frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}\partial_\mu A_\nu B_{\lambda\rho} = \frac{1}{2}\varepsilon^{ijk}(\dot{A}_i B_{jk} + A_0\partial_i B_{jk} + 2B_{0k}\partial_i A_j) \tag{12}$$

the fields A_i and $\frac{1}{2}\varepsilon^{ijk}B_{jk}$ are already canonical conjugates:

$$\left[A_k(x), \frac{1}{2}\varepsilon^{lij}B_{ij}(y)\right] = i\delta^l_k\delta^3(x-y) \tag{13}$$

Then, the Hamiltonian results to be:

$$H = \int d^3x \left\{\frac{1}{2}(\pi^i_C)^2 + \frac{1}{4}(F_{ij}(C) + B_{ij})^2 - C_0(\pi^i_{C,i}) - \frac{1}{2}A_0\varepsilon^{ijk}B_{jk,i} + B_{k0}(\pi^k_C - \varepsilon^{ijk}A_{i,j})\right\} \tag{14}$$

where $\pi^i_C(x)$ are the momenta conjugate to $C_i(x)$.

The primary constraints:

$$\pi_{A_0} \approx 0 \quad , \quad \pi_{C_0} \approx 0 \quad , \quad \pi_{B_{K0}} \approx 0 \tag{15}$$

lead to the secondary ones:

$$\pi^i_{c,i} \approx 0 \tag{16}$$

$$\varepsilon^{ijk} B_{jk,i} \approx 0 \tag{17}$$

$$\pi^k_c \approx \varepsilon^{ijk} A_{i,j} \tag{18}$$

as is easily verified. Since constraint (16) is consequence of (18), the former may be ignored from now on. It can be checked that all the constraints are first class, as correspond to a "full" gauge invariant theory. When acting over physical states the Hamiltonian reduces to:

$$H = \int d^3x \frac{1}{2} \left\{ \mathcal{B}^i \mathcal{B}^i + \mathcal{E}^i \mathcal{E}^i \right\} \tag{19}$$

where we have set:

$$\mathcal{B}^k \equiv \varepsilon^{ijk} A_{i,j} \tag{20}$$

$$\mathcal{E}^i \equiv \frac{1}{2} \varepsilon^{ijk} (B_{jk} + F_{jk}(c)) \tag{21}$$

Operators $\mathcal{E}^i$ and $\mathcal{B}^i$ obey the algebra of the electric and magnetic operators of Maxwell theory, as expected.

Now we proceed with the geometric representation appropiate to the formulation we are dealing with. The secondary first class constraints (17), (18) generate, as one could expect, the time independent gauge transformations that correspond to equations (9)-(11). Within the spirit of the L.R., we seek for non-local gauge invariant operators that replace the gauge dependent canonical

ones A_i, C_i and B_{ij}. An adecuate choice consists on the Wilson loop:

$$W(\gamma) = \exp(i \oint_\gamma dy^i A_i(y)) \tag{22}$$

where γ is a closed spatial path, and the operator:

$$\Omega(\Sigma, \Gamma) = \exp(i \oint_\Gamma dy^i C_i(y)) \exp(i \int\int_\Sigma d\Sigma_k \varepsilon^{ijk} B_{ij}) \tag{23}$$

which depends on the spatial open surface Σ whose boundary is Γ. Taking an infinitesimal loop $\delta\gamma$ and surface $\delta\Sigma$ one has:

$$W(\delta\gamma) = \mathbf{1} + i\delta\sigma^{ij} F_{ij}(A) + \theta(\delta\sigma^2) \tag{24}$$

$$\Omega(\delta\Sigma, \delta\Gamma) = \mathbf{1} + i\delta\Sigma^{ij}(F_{ij}(c) + B_{ij}) + \theta(\delta\Sigma^2) \tag{25}$$

which shows that W and Ω encode the local gauge invariant content of A_i, C_i and B_{ij}, as claimed.

From the canonical commutators, it is easy to calculate the algebra obeyed by W and Ω:

$$W(\gamma)\Omega(\Sigma, \Gamma) = e^{i\mathcal{L}(\gamma,\Gamma)} \Omega(\Sigma, \Gamma) w(\gamma) \tag{26}$$

where:

$$\begin{aligned} \mathcal{L}(\gamma, \Gamma) &= \oint_\gamma dx^i \int\int_\Sigma d\Sigma_i(y) \delta^3(x-y) \\ &= \frac{1}{4\pi} \oint_\gamma dx^i \oint_\Gamma dy^j \varepsilon_{ijk} \frac{(x-y)^k}{|x-y|^3} \end{aligned} \tag{27}$$

is the linking number between the loops γ, Γ. Equation (26), which we shall call dual algebra (D.A.) of Maxwell theory, deserves the following comments. In virtue of constraints (17) one has that on the physical (i.e., gauge invariant) subspace:

$$\Omega(\Sigma_{closed})|\Psi_{physical} >= |\Psi_{physical} > \tag{28}$$

that is to say, Ω does not depend on Σ, but only on its boundary Γ. Thus, $\Omega(\Gamma, \Sigma)$, which by construction is the quantum operator associated to the electric flux through Σ, can also be viewed as the "dual" Wilson Loop, i.e., as the contour integral of the dual potential $\widetilde{A}$ along Γ. Observe, however, that the formulation does not include this potential as a lagrangian variable, which would be redundant.

It should be noticed the topological character of the D.A.. Both W and Ω, inasmuch as the linking number $\mathcal{L}(\gamma, \Gamma)$ are metric independent objects. The latter is, in fact, a "link invariant", in knot theoretical terms. Of course, this does not mean that Maxwell theory is a topological one, since the Hamiltonian is constructed from the metric dependent combinations $\mathcal{E}^i\mathcal{E}^i$ and $\mathcal{B}^i\mathcal{B}^i$.

The algebra (26) may be realized by prescribing that W and Ω act onto loop dependent wave functionals $\Psi(\gamma)$ as:

$$W(\gamma_2)\Psi(\gamma_1) = \Psi(\gamma_2 \cdot \gamma_1) \tag{29}$$

and

$$\Omega(\Gamma)\Psi(\gamma_1) = e^{-i\mathcal{L}(\Gamma,\gamma_1)}\Psi(\gamma_1) \tag{30}$$

Here, $\Psi(\gamma) \equiv <\gamma|\Psi>$ is the wave functional in the usual L.R. $|\gamma>$ [12], while $\gamma_1 \cdot \gamma_2$ denotes the Abelian Group of Loops product [12]. To check that (29) (30) realize the D.A. (26) we have used the following elementary properties of the linking number:

$$\mathcal{L}(\gamma_1, \gamma_2) = \mathcal{L}(\gamma_2, \gamma_1) \tag{31}$$

$$\mathcal{L}(\gamma_1, \gamma_2 \cdot \gamma_3) = \mathcal{L}(\gamma_1, \gamma_2) + \mathcal{L}(\gamma_1, \gamma_3) \tag{32}$$

Equations (29), (30) have a simple geometrical meaning: $W(\gamma)$ performs a finite translation in Loop space, while $\Omega(\Gamma)$ (we suppress the spurious Σ-dependence) acts multiplicatively, by

measuring how Γ links with the wave functional argument. These roles are exchanged if one choose to work in the dual L.R. $|\Gamma>$: from equation (26), we see that the replacements $W(\gamma)\leftrightarrows\Omega(\Gamma)$ and $\mathcal{L}(\gamma,\Gamma) \to -\mathcal{L}(\gamma,\Gamma)$ leave the D.A. invariant. This is the way how self-duality of $D=4$ Maxwell theory is reflected in the present formulation.

It is worth mentioning that the D.A. (26) is closely related to the well known 't Hooft algebra, [10] which was obtained in the context of pure Yang-Mills theory. However, although the 't Hooft analogous to our $\Omega(\Gamma)$ (the B operator in 't Hooft notation) can be interpreted as measuring in some sense the chromo-electric flux, it is not possible, as far as we know, to write it down in terms of the chromo-electric field. Hence, unlike the present case, W and B cannot describe the phase space, nor the Hamiltonian can be written in terms of them. This feature is related to the fact that there not exists a simple non-Abelian analogous to electric-magnetic duality [7,8].

To complete the description of quantum Maxwell theory within the above formulation, we have to express H in terms of W and Ω. To this end, we recall the loop derivative of Gambini-Trías [12]:

$$\Delta_{ij}(x)f(\gamma) = \lim_{\sigma^{ij}\to 0} \frac{f(\delta\gamma\cdot\gamma) - f(\gamma)}{\sigma^{ij}} \tag{33}$$

that measures how a loop-dependent object $f(\gamma)$ changes when γ is modified by appending a small contour $\delta\gamma$ of area σ^{ij} at x. Then the electric and magnetic field operators are given by:

$$\mathcal{B}^k(x) = -i\varepsilon^{ijk}\Delta_{ij}(x)w(\gamma)|_{\gamma=0} \tag{34}$$

$$\mathcal{E}^k(x) = -i\varepsilon^{ijk}\Delta_{ij}(x)\Omega(\gamma)|_{\Gamma=0} \tag{35}$$

and a loop-dependent Schrödinger equation can be written down, either in the $|\gamma>$ or the $|\Gamma>$ loop representations.

To conclude, let us briefly sketch how this ideas may be generalized to abelian gauge theories of arbitrary rank and dimension. To save some writing, we turn to the more compact form-notation. Consider, in $D = d+1$ dimensions, the theory associated to a p-form A ($p < D$) whose lagrangian is given by:

$$\mathcal{L}(A) = \alpha dA \wedge^* (dA) \tag{36}$$

As it is well known, the dual theory that correspond to (36) is described by the lagrangian $\mathcal{L}(\widetilde{A})$, where $\widetilde{A}$ is a $D-(p+2)$ form. It follows that for even D, self-dual theories are constructed from $\frac{D-2}{2}$ forms. The first order gauge-invariant lagrangian $\mathcal{L}''$ (see equation (8)) associated to (36) is:

$$\mathcal{L}'' = \alpha dA \wedge B + \beta(B + dC) \wedge^* (B + dC) \tag{37}$$

where B and C are $D-(p+1)$ and $D-(p+2)$ forms respectively. $\mathcal{L}''$ is invariant under:

$$\delta A = d\wedge \tag{38.a}$$

$$\delta B = d\xi \tag{38.b}$$

$$\delta C = \xi \tag{38.c}$$

As in Maxwell case, the dinamical equations obtained from $\mathcal{L}$ and $\mathcal{L}''$ are equivalent.

Canonical quantization follows along the same lines as before. Our main concern, which is to show the D.A. appropiate to the theories we are dealing with, may be resumed as follows. The first class constraints that arise tell us that the non-local operators carrying the gauge invariant content of the pull-backs of A, B and C to the spatial manifold $\mathcal{R}^d$, can be taken as:

$$W(S_a) = \exp(i \int_{S_a} A) \tag{39}$$

$$\Omega(S_b) = \exp(i \int_{S_b} (B + dC)) \tag{40}$$

where S_a is a closed p-surface embeded in $\mathcal{R}^d$, while S_b is an open $d-p$ one. As before, one of the constraints (the pull-back of $dB = 0$) allows to show that on the physical subspace, Ω only depends on the boundary ∂S_b of S_b. Then the D.A. may be written as:

$$W(S_a)\Omega(\partial S_b) = e^{i\mathcal{L}(S_a,\partial S_b)}\Omega(\partial S_b)W(S_a) \tag{41}$$

where:

$$\mathcal{L}(S_a, \partial S_b) \equiv \frac{1}{k}\int_{S_a} d\Sigma^{i_1\cdots i_p}(x)\int_{\partial S_b} d\Sigma^{i_{p+2}\cdots i_d}\frac{(x-y)^{i_{p+1}}}{|x-y|^d}\varepsilon_{i_1\cdots i_d} \tag{42}$$

measures the linking number of S_a and ∂S_b (k is the area of the unit d-sphere). Equation (42) does not include the cases where either A, B, or both two are 0-forms. To seek for the appropriate expressions for that cases, it must be observed that the boundary of a 1-dimensional "open surface" (i.e., a open path) is a pair of oriented points (x, y). Thus, when A is a 0-form $\varphi(x)$, its associated $W(S_a)$ will be:

$$W((x,y)) = \exp i \{\varphi(x) - \varphi(y)\} \tag{43}$$

and the correponding linking number for the case $A = \varphi(x)$, $d \geq 2$ is:

$$\mathcal{L}((x,y), \partial S_b) = \frac{1}{k}\int_{\partial S_b} d\Sigma^{i_1\cdots i_{d-1}}_{(z)}(z)\varepsilon_{i_1\cdots i_d}\left\{\frac{(z-y)^{i_d}}{|x-y|^d} - \frac{(z-x)^{i_d}}{|x-y|^d}\right\} \tag{44}$$

This expression counts how many times the $d-1$ surface ∂S_b encloses the points x and y, taking into account that they contribute with opposite signs.

On the other hand, for $d = 1$, where A is forced to be a 0-form, both W and Ω result to be operators depending on pairs of points. The corresponding D.A. is given by:

$$W((x,y))\Omega((x',y')) = \exp i \{\theta(x-x') - \theta(x-y') - \theta(y-x') + \theta(y-y')\} \tag{45}$$

The combination of Heaviside functions θ appearing in (45) is a topological invariant associated to the pairs (x, y), (x', y'): it measures whether or not the line segment that corresponds to one of the pairs contains some point of the other pair. This quantity is the appropriate "linking number" for points on a line.

Summarizing, we have exploited the geometrical language of the L.R. to show that the "electric magnetic" duality of usual abelian gauge theories leads in a natural manner to a topological algebra of elementary observables that may replace the gauge-dependent canonical algebra. This algebra admits a realization in terms of operators acting on functionals that depend on geometrical objects, which is explicitly shown for the four dimensional Maxwell Theory.

The above ideas can be extended without mayor difficulties to the case of massive theories (i.e., Proca's model and its higher rank generalizations). It remains to be studied whether or not the present formulation can be useful to explore the analogous to electric-magnetic duality in non Abelian theories.

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